# SOLUTIONS OF THE EQUATIONS OF A LAMINAR BOUNDARY LAYER FOR SMALL TEMPERATURE FACTORS AND HIGH MACH NUMBERS

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The equations for a gradient-free laminar boundary layer have been integrated numerically on a computer for small temperature factors $\bar{T}_F$ and high Mach numbers M. It is shown that a calculation of the heat-transfer and friction coefficients by the "controlling-temperature method" leads to a good agreement with the coefficients found from the exact solutions of the equation of the boundary layer with and without injection.

In the many papers which have been published on the boundary layer, a variety of assumptions have been used regarding the way to take into account the influence of the temperature factor $\bar{T}_F$ on the heat transfer and the friction. In [1, 2] and in certain other papers, the influence of $\bar{T}_F$ on the heat transfer of a laminar boundary layer is taken into account by introducing a "controlling" (reduced) temperature in the dimensionless equation for the incompressible fluid. The influence of the temperature factor (for a laminar boundary layer) is studied analytically in [1] for Le and Pr = 1 with $T_w/T_e \gg 1$ or $0.2\,M^2$. For the case $\varkappa = 1.4$ it was found there that T* should be calculated from

$$T^* = T_e\left(0.5 + 0.033M_e^2 + 0.5\frac{T_w}{T_e}\right). \tag{1}$$

The equation for the "controlling" temperature T* was not studied analytically in [2]. It was recommended there that T* be calculated from

$$T^* = (T_w - T_e)\,0.5 + T_e + 0.22\,(T_r - T_e). \tag{2}$$

For practical purposes we are interested in the influence of the temperature factor $T_F$ on the solutions of the boundary-layer equation for $T_F \ll 1$ and for large numbers M. Here we consider the case Pr = Le = 1, dp/dx = 0, and $T_w = T_e$.

The initial equations are as follows:

the flow rate equation,

$$[\rho u]_x + [\rho v]_y = 0; \tag{3}$$

the momentum equation,

$$\rho u u_x + \rho v u_y = [\mu u_y]_y = \tau_y; \tag{4}$$

the equation of state,

$$\rho = \frac{p_e}{gRT}; \tag{5}$$

the equation for the dynamic viscosity,

$$\mu = kT^{0.5}; \tag{6}$$

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and the energy equation,

$$\rho u T_{0x} + \rho v T_{0y} = [\mu T_{0y}]_y. \tag{7}$$

Here $T_0$ is the stagnation temperature,

$$T_0 = T + \frac{u^2}{2gc_pA}. \tag{8}$$

System (3)-(8) is solved under the following boundary conditions:

$$\begin{aligned} &y=0; \quad u=0; \quad v=v_w; \quad T=T_w=T_{0w}; \\ &y\to\infty; \quad u\to u_e; \quad T\to T_e; \quad u_{ye}\to 0. \end{aligned} \tag{9}$$

If $T_W$ is not a function of x, then the solution of Eq. (7) is the Crocco integral [1]

$$T_0 = \frac{T_{0e}-T_w}{u_e}u + T_w, \tag{10}$$

where

$$T_{0e} = T_e + \frac{u_e^2}{2gc_pA}. \tag{11}$$

We find the temperature from Eq. (10):

$$T = \frac{T_{0e}-T_w}{u_e}u + T_w - \frac{u^2}{2gc_pA}. \tag{12}$$

We restrict the present analysis to the case $T_W = T_e$: then using (11) we find from (12)

$$T = \frac{uu_e - u^2}{2gc_pA} + T_w. \tag{13}$$

Through a substitution of variables, we can transform the partial differential equations in (3), (4), and (7) into ordinary differential equations. We seek solutions u and v in the form

$$u = u_e f\left(\frac{\bar{y}}{\sqrt{x}}\sqrt{\mathrm{Re}_u}\right); \quad v = \frac{u_e}{\sqrt{x\,\mathrm{Re}_u}}\varphi\left(\frac{\bar{y}}{\sqrt{x}}\sqrt{\mathrm{Re}_u}\right). \tag{14}$$

After some substitutions and manipulations involving Eqs. (13) and (14), the flow-rate equation (3) becomes

$$-\frac{1}{2}\bar{\rho}_f ff'\theta - \frac{1}{2}\bar{\rho}\theta f' + \bar{\rho}_f\varphi f' + \varphi'\bar{\rho} = 0, \tag{15}$$

and the momentum equation (4) becomes

$$\bar{\rho}\varphi f' - \frac{1}{2}ff'\bar{\rho}\theta = z', \tag{16}$$

where $z = \bar{\mu}f'$.

System (15), (16) is a closed system of equations. We note that substitution (14) is quite convenient for a solution of this system of equations on a computer. For system (15), (16) we find the following boundary conditions, instead of those in (9):

$$\theta = 0; \quad f_w = 0; \quad \varphi = \varphi_w; \tag{17}$$

$$\theta \to \infty; \quad f_e \to 1; \quad f'_e \to 0. \tag{18}$$

System (15), (16) is integrated by the very simple Euler–Cauchy method. The integration step is chosen such that a further reduction of the step does not affect the results of the solution. The relationship between the boundary conditions at the wall and at the outer boundary of the boundary layer is found by the method of trial and error; specifically, the derivative $f'_W$ (at the wall) is specified in the computer program in a manner such that boundary conditions (18) are satisfied. We use this program to calculate the velocity profile, the

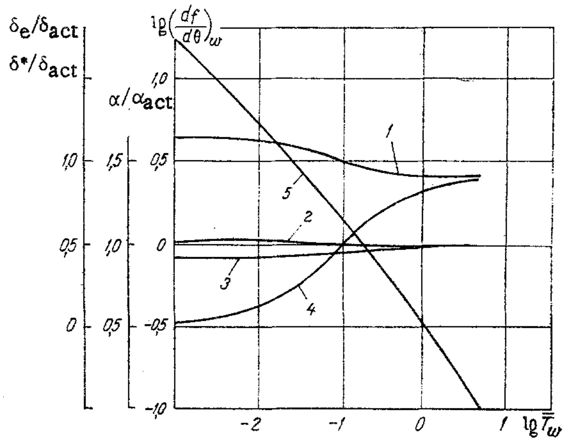


Fig. 1. Properties of the boundary layer as functions of $\overline{T}_W$ (B = 0): 1) $\delta^*/\delta_{act}$ (T* is calculated by the Dorrance method); 2) $\alpha^*/\alpha_{act}$ (T* is calculated by the Dorrance method); 3) $\alpha^*/\alpha_{act}$ (T* is calculated by the Eckert–Drake method); 4) $\delta_e/\delta_{act}$; 5) $\log(df/d\theta)_W$.

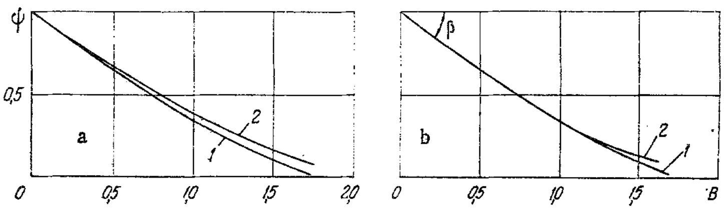


Fig. 2. Effect of injection on the heat-transfer coefficient upon a change in $\overline{T}_W$. a) T* (calculated by the Eckert–Drake method); b) T* (calculated by the Dorrance method). 1) $T_W = 1.0$; 2) $T_W = 0.001$. The values of $\psi$ for $1.0 > \overline{T}_W > 0.001$ lie between the curves of $\psi$ for $T_W = 1.0$ and $\overline{T}_W = 0.001$.

heat-transfer coefficient, and the boundary-layer thickness over a broad range $\overline{T}_W = 0.001$–$5$. At small values $\overline{T}_W$, the heat transfer and the friction can, of course, be substantially affected by the changes in the gas properties due to the high temperatures, chemical reactions, radiative heat transfer, etc. However, these effects are of no importance to the problem of the present paper.

Let us convert the equations for the controlling temperature, (1) and (2), to a form convenient for analysis in the case $T_W = T_e$, Pr = 1. We express $M_e$ in (1) in terms of $u_e$ and the sound velocity. From (1) we find

$$T^* = T_w + \frac{0.165u_e^2}{2gc_pA}\,; \tag{19}$$

from (2) we find

$$T^* = T_w + \frac{0.22u_e^2}{2gc_pA}\,. \tag{20}$$

I. Analysis of the Characteristics of a Thermal Boundary Layer in the Absence of Injection. We return to boundary conditions (17). In the case of no injection we have $\varphi_W = 0$. Figure 1 shows the change in the ratio of the heat-transfer coefficients calculated by the controlling-temperature method to the heat-transfer coefficient calculated by the program of the present work (we note that the ratio of heat-transfer coefficients is equal to the ratio of friction coefficients). We see from Fig. 1 that if T* is calculated from Eq. (20), we find $1 > \alpha^*/\alpha_{act} > 0.94$; if we instead calculate T* from Eq. (19), we find $\alpha^*/\alpha_{act} = 1:1.013$ over the entire range of $\overline{T}_W$. Accordingly, there is a solid basis for assuming that the controlling-temperature method involving the use of Eq. (19) to calculate T* can be recommended for calculations of heat transfer and friction at small values of

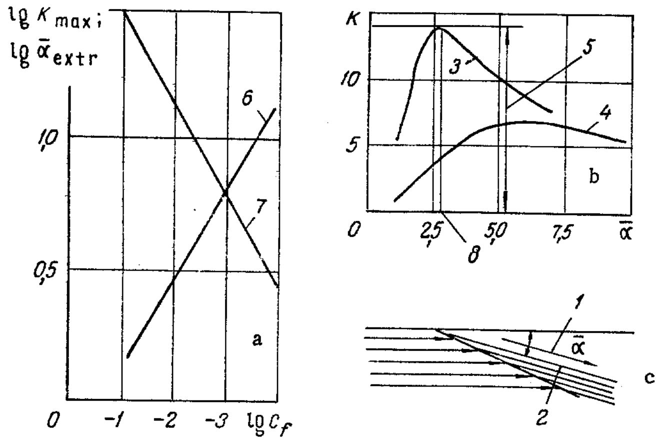


Fig. 3. a) Maximum quality and angle of attack $\bar{\alpha}_{extr}$ of the plate as functions of $C_f$; b) aerodynamic quality of the plate as a function of the angle of attack; c) plate in hypersonic flow. 1) Friction force $x_{fr} = C_f \cdot (\rho_i u_i^2/2)F$; 2) pressure force $P = (\rho_i u_i^2/2)F$; 3) $C_f = 0.0001$; 4) $C_f = 0.001$; 5) $K_{max}$; 6) $\log K_{max}$; 7) $\log \bar{\alpha}_{extr}$; 8) $\bar{\alpha}_{extr}$.

$\bar{T}_w$ for laminar flow with Pr = Le = 1, dp/dx = 0, and $T_w = T_e$. We know that for an incompressible laminar boundary layer the thickness of the dynamic boundary layer is calculated from

$$\delta_{in} = \frac{4.5L}{\mathrm{Re}^{0.5}}. \tag{21}$$

For a compressible velocity boundary layer there are no definite recommendations for calculating the thickness of a boundary layer for a small temperature factor. Let us determine which of the following equations should be used to calculate the thickness of a compressible boundary layer:

$$\delta^* = \frac{4.5L}{\mathrm{Re}^{*0.5}}, \tag{22}$$

or

$$\delta_e = \frac{4.5L}{\mathrm{Re}_e}. \tag{23}$$

Figure 1 also shows the calculated values of the ratios $\delta_e/\delta_{act}$ and $\delta^*/\delta_{act}$, where $\delta_{act}$ is the actual thickness of the boundary layer. In the determination of $\delta_{act}$ at the boundary of the boundary layer we assumed $u = 0.99u_e$. The data in Fig. 1 show that $\delta^*/\delta_{act}$ lies in the range 0.92-1.15 over the entire range of $\bar{T}_w$; i.e., within an error of 13% the thickness of the boundary layer can be calculated from Eq. (22). We see from Fig. 1 that the boundary-layer thickness calculated from Eq. (23) is completely unacceptable.

II. Analysis of the Influence of Injection on Heat Transfer and Friction over Broad Ranges of $\bar{T}_w$ and M. Let us examine boundary conditions (17). In the case of injection we have $\varphi_w = \mathrm{const} \neq 0$. The calculated results are treated in the form of curves of $\psi = f(B)$, where

$$\psi = \frac{\alpha}{\alpha_{B=0}} = \frac{\tau_w}{\tau_{wB=0}} = \frac{\mathrm{St}^*}{\mathrm{St}^*_{B=0}}; \quad B = \frac{\rho_w v_w}{\rho^* u_e \mathrm{St}^*_{B=0}}. \tag{24}$$

We note that for an incompressible boundary layer the results of the calculation of the heat transfer during injection are treated in the form of the following function [1]:

$$\frac{\alpha}{\alpha_{B=0}} = \frac{\mathrm{St}_e}{\mathrm{St}_{eB=0}} = f(B_{in}); \quad B_{in} = \frac{\rho_w v_w}{\rho_e u_e \mathrm{St}_e}. \tag{25}$$

We see that Eq. (24) differs from (25) in that the parameters $\rho^*$ and $\mathrm{St}^*_{B=0}$ in (24) are calculated at the controlling temperature. Figure 2 shows the calculated results with injection taken into account. These results show that when T* is calculated from Eq. (19) the results of the exact calculation for B < 1.4 essentially conform to a common universal curve, whose slope at small values of (B < 0.3) is the same as the slope of the curve obtained for an incompressible laminar boundary layer ($\tan\beta \cong 2/3$) [1]. If the injection parameter is calculated at T* as calculated from (20), we do not find a single universal curve (curve 2 in Fig. 2a).

Engineering Application. Examination of the data in Fig. 2 and of the theoretical equations above leads to the following conclusions.

1. Injection is effective at large Mach numbers M and at low temperature factors $\overline{T}_w$.

2. Intense injection substantially reduces both heat transfer and friction, and these effects may turn out to be useful in the design of certain hypersonic aircraft. Let us illustrate this latter assertion. We consider a plate oriented at some angle of attack in a hypersonic flow (Fig. 3c). The aerodynamic quality K of this plate can be written as

$$K = \frac{\overline{p}\cos\overline{\alpha} - C_f \sin\overline{\alpha}}{\overline{p}\sin\overline{\alpha} + C_f \cos\overline{\alpha}},$$

where $C_f$ is the coefficient of friction averaged over the plate, divided by the incoming velocity head $\rho_i u_i^2/2$. We assume that the pressure coefficient $\overline{p}$ obeys the Newton law $\overline{p} = 2\sin^2\overline{\alpha}$:

$$K = \frac{2\sin^2\overline{\alpha}\cos\overline{\alpha} - C_f \sin\overline{\alpha}}{2\sin^3\overline{\alpha} + C_f \cos\overline{\alpha}}. \tag{26}$$

The results of a calculation carried out to determine the behavior of K as a function of $\overline{\alpha}$ for a constant coefficient $C_f$ are shown in Fig. 3b. The data of this figure show that the curve $K = K(\overline{\alpha})$ has a maximum $K_{max}$.

Figure 3a shows curves of $K_{max}$, $\overline{\alpha}_{extr}$ as functions of $C_f$. We see that the aerodynamic quality $K_{max}$ can be substantially increased by reducing $C_f$. Intense injection reduces $C_f$, and the plate becomes "slippery." Accordingly, in certain cases, intense injection may turn out to be useful for hypersonic aircraft.

## NOTATION

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| $u$ | is the horizontal velocity component; |
| $v$ | is the vertical velocity component; |
| $\tau$ | is the frictional stress; |
| $x, y$ | are the coordinates; |
| $\rho$ | is the density; |
| $p$ | is the pressure; |
| $M_e$ | is the Mach number at the outer boundary of the boundary layer; |
| $\mu$ | is the dynamic viscosity; |
| $g$ | is the acceleration due to gravity; |
| $R$ | is the universal gas constant; |
| $c_p$ | is the specific heat at p = const; |
| $A$ | is the coefficient for the conversion of heat into mechanical work; |
| $\varkappa = c_p/c_v$ | is the ratio of specific heats; |
| $T$ | is the temperature; |
| $T_0$ | is the stagnation temperature; |
| $k$ | is the coefficient in the equation for the dynamic viscosity; |
| Pr | is the Prandtl number; |
| Le | is the Lewis number; |
| Re | is the Reynolds number; |
| St | is the Stanton number; |
| $L$ | is the scale dimension; |
| $T_u = u_e^2/2gc_pA$; $Re_u = p_e u_e L/gRT_u kT_u^{0.5}$; $Re^* = \rho^* u_e L/\mu^*$; $Re_e = \rho_e u_e L/\mu_e$; $\overline{T}_F = T_w/T_0$ | is the temperature factor; |
| $\overline{T}_w = T_w/T_u$; $\overline{\rho} = 1/(f - f^2 + \overline{T}_w)$; $d\overline{\rho}/df = \overline{\rho}f = (2f-1)/(f-f^2+\overline{T}_w)^2$; $\overline{\mu} = (f-f^2+\overline{T}_w)^{0.5}$; $\overline{x} = x/L$; $\overline{y} = y/L$; $\theta = (\overline{y}/\sqrt{\overline{x}})\sqrt{Re_u}$ | is the self-similar variable; |
| $f' = df/d\theta$; $\varphi' = d\varphi/d\theta$; $T_r$ | is the reduction temperature; |

$\partial(\rho u)/\partial x = [\rho u]_x$; $\partial(\rho v)/\partial y = [\rho v]_y$;
$\partial u/\partial x = u_x$; $\bar{p} = (p - p_i)/(\rho_i u_i^2/2)$

Indices

w — is the properties at walls:
e — is the properties at outer boundary of boundary layer;
act — is the properties obtained through the solution of the boundary-layer equation;
* — is the properties calculated on the basis of the controlling temperature;
i — is the properties of incoming flow;
in — is the incompressible fluid.

## LITERATURE CITED

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# QUASISTEADY APPROACH IN CALCULATIONS FOR CONVECTIVE HEAT TRANSFER

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An equation is derived for determining the limits of applicability of the quasisteady approach in thermal calculations involving the cooling of metal plates in liquids and gases.

Under certain conditions of convective heat transfer, a boundary layer rapidly reacts to external perturbations and manages to change when there are changes in either the temperature of the object in the flow, the pressure at the inlet to the channel, or other parameters. In this case, "instantaneous steady states" exist at each time, and the steady-state approach can be used to determine the rate of the process. We call processes occurring under such conditions "quasisteady."

Whether a particular type of heat transfer can be treated as quasisteady is of both theoretical and practical interest. The use of equations found for the steady-state conditions substantially simplifies the calculations. There has been less study of unsteady heat-transfer processes, and dimensionless equations for the heat-transfer coefficients are not available for most unsteady processes.

The usual approach is to treat a heat-transfer process as quasisteady if the ratio of the Nusselt numbers found experimentally and calculated on the basis of the equations corresponding to the steady-state regimes is approximately unity [1-3]. In certain cases, the "condition for a quasisteady system" is assumed to be the approximate equality of the steady and unsteady heat fluxes [4, 5].

Attempts have been made to find the conditions under which the equations found for the steady-state conditions can be applied to unsteady heat-transfer processes. Comparing the heat fluxes calculated for the steady and unsteady regimes during the heating and cooling of a vertical plate, Sparrow and Gregg [4] found that the process can be treated as quasisteady under the condition

$$\frac{\dot{\Delta T}}{\Delta T}\left[\frac{xT_\infty}{g(\Delta T)}\right]^{1/2} \leqslant 0.033,$$

where

$$\Delta T = T - T_\infty; \quad \dot{\Delta T} = \frac{d(\Delta T)}{d\tau}.$$

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